LOCAL GEODETIC HORIZON COORDINATES

In many surveying applications it is necessary to convert *geocentric* Cartesian coordinates X,Y,Z to *local geodetic horizon* Cartesian coordinates E,N,U (East,North,Up). Figure 1 shows a portion of a reference ellipsoid (defined by semi-major axis a and flattening f) approximating the size and shape of the Earth. The origin of the X,Y,Z coordinates lies at O, the centre of the ellipsoid (assumed to be the Earth's centre of mass, hence the name Geocentric). The *Z-axis* is coincident with the Earth's rotational axis and the X-Z plane is the Greenwich meridian plane (the origin of longitudes λ). The X-Y plane coincides with the Earth's equatorial plane (the origin of latitudes ϕ) and the positive *X-axis* is in the direction of the intersection of the Greenwich meridian plane and the equatorial plane. The positive *Y-axis* is advanced 90° east along the equator.

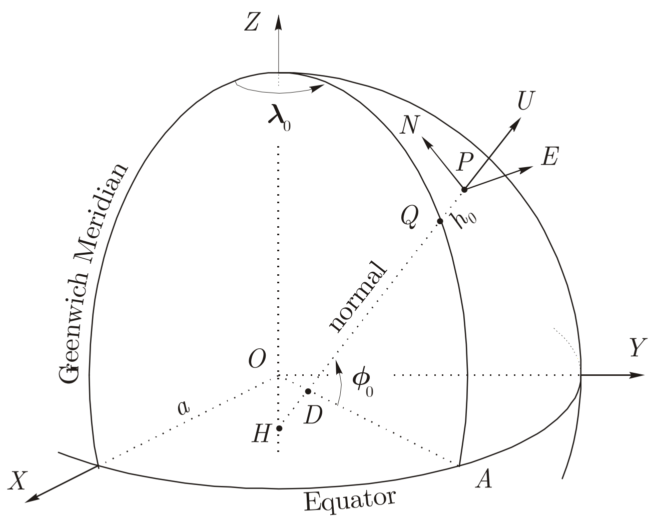


Figure 1 Geocentric and Local coordinate axes and the reference ellipsoid

A point P on the Earth's terrestrial surface is referenced to the ellipsoid via the normal that passes through P and intersects the ellipsoid at Q. The normal through P intersects the equatorial plane at D and cuts the *Z-axis* at H. The angle between the normal and the equatorial plane is the latitude ϕ (0° to 90° positive north, negative south). The height of the point above the ellipsoid (measured along the normal) is the ellipsoidal height h.

The angle between the Greenwich meridian plane and the meridian plane of the point (the plane containing the normal and the *Z-axis*) is the longitude λ (0° to 180° positive east, negative west). Geocentric Cartesian coordinates are computed from the following equations

$$\begin{aligned} X &= (\nu + h)\cos\phi\cos\lambda \\ Y &= (\nu + h)\cos\phi\sin\lambda \\ Z &= \left(\nu\left(1-e^2\right)+h\right)\sin\phi \end{aligned} \tag{1}$$

where $\nu = \dfrac{a}{\sqrt{1-e^2\sin^2\phi}}$ is the radius of curvature of the ellipsoid in the prime vertical plane. In Figure 1, $\nu = QH$

$e^2 = f(2-f)$ is the square of the eccentricity of the ellipsoid.

The origin of the E,N,U system lies at the point $P(\phi_0, \lambda_0, h_0)$. The positive *U-axis* is coincident with the normal to the ellipsoid passing through P and in the direction of increasing ellipsoidal height. The N-U plane lies in the meridian plane passing through P and the positive *N-axis* points in the direction of North. The E-U plane is perpendicular to the N-U plane and the positive *E-axis* points East. The E-N plane is often referred to as the *local geodetic horizon plane.*

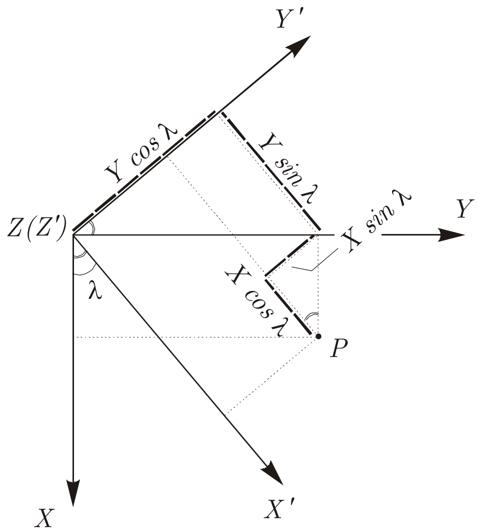
Geocentric and local Cartesian coordinates are related by the matrix equation

$$\begin{bmatrix} U \\ E \\ N \end{bmatrix} = \mathbf{R}_{\phi\lambda}\begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix} \tag{2}$$

where X_0, Y_0, Z_0 are the geocentric Cartesian coordinates of the origin of the E,N,U system and $\mathbf{R}_{\phi\lambda}$ is a rotation matrix derived from the product of two separate rotation matrices.

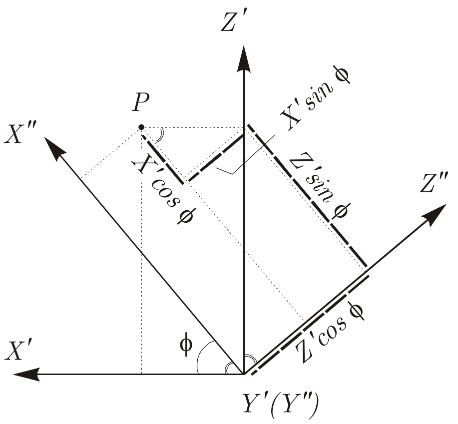
$$\mathbf{R}_{\phi\lambda} = \mathbf{R}_{\phi}\mathbf{R}_{\lambda} = \begin{bmatrix} \cos\phi_0 & 0 & \sin\phi_0 \\ 0 & 1 & 0 \\ -\sin\phi_0 & 0 & \cos\phi_0 \end{bmatrix}\begin{bmatrix} \cos\lambda_0 & \sin\lambda_0 & 0 \\ -\sin\lambda_0 & \cos\lambda_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3}$$

The first, $\mathbf{R}_\lambda$ (a positive right-handed rotation about the *Z-axis* by λ) takes the *X, Y, Z* axes to X', Y', Z'. The Z' *-axis* is coincident with the *Z-axis* and the $X' - Y'$ plane is the Earth's equatorial plane. The $X' - Z'$ plane is the meridian plane passing through P and the Y' *-axis* is perpendicular to the meridian plane and in the direction of East.



$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\lambda & \sin\lambda & 0 \\ -\sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{R}_\lambda} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

The second $\mathbf{R}_\phi$ (a rotation about the *Y′-axis* by ϕ) takes the X', Y', Z' axes to the X'', Y'', Z'' axes. The $X'' - axis$ is parallel to the *U-axis,* the $Y'' - axis$ is parallel to the *E-axis* and the $Z'' - axis$ is parallel to the *N-axis.*



$$\begin{bmatrix} X'' \\ Y'' \\ Z'' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}}_{\mathbf{R}_\phi} \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix}$$

Performing the matrix multiplication in equation (3) gives

$$\mathbf{R}_{\phi\lambda} = \begin{bmatrix} \cos\phi_0 \cos\lambda_0 & \cos\phi_0 \sin\lambda_0 & \sin\phi_0 \\ -\sin\lambda_0 & \cos\lambda_0 & 0 \\ -\sin\phi_0 \cos\lambda_0 & -\sin\phi_0 \sin\lambda_0 & \cos\phi_0 \end{bmatrix} \tag{4}$$

Rotation matrices formed from rotations about coordinate axes are often called Euler rotation matrices in honour of the Swiss mathematician Léonard Euler (1707-1783). They are orthogonal, satisfying the condition $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ (i.e., $\mathbf{R}^{-1} = \mathbf{R}^T$).

A re-ordering of the rows of the matrix $\mathbf{R}_{\phi\lambda}$ gives the transformation in the more usual form E, N, U

$$\begin{bmatrix} E \\ N \\ U \end{bmatrix} = \mathbf{R} \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix} \tag{5}$$

where

$$\mathbf{R} = \begin{bmatrix} -\sin\lambda_0 & \cos\lambda_0 & 0 \\ -\sin\phi_0 \cos\lambda_0 & -\sin\phi_0 \sin\lambda_0 & \cos\phi_0 \\ \cos\phi_0 \cos\lambda_0 & \cos\phi_0 \sin\lambda_0 & \sin\phi_0 \end{bmatrix} \tag{6}$$

From equation (5) we can see that coordinate differences $\Delta E = E_k - E_i$, $\Delta N = N_k - N_i$ and $\Delta U = U_k - U_i$ in the *local geodetic horizon plane* are given by

$$\begin{bmatrix} \Delta E \\ \Delta N \\ \Delta U \end{bmatrix} = \mathbf{R} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} \tag{7}$$

where $\Delta X = X_k - X_i$, $\Delta Y = Y_k - Y_i$ and $\Delta Z = Z_k - Z_i$ are geocentric Cartesian coordinate differences.

NORMAL SECTION AZIMUTH ON THE ELLIPSOID

The matrix relationship given by equation (7) can be used to derive an expression for the azimuth of a normal section between two points on the reference ellipsoid. The normal section plane between points P_1 and P_2 on the Earth's terrestrial surface contains the normal at point P_1, the intersection of the normal and the rotational axis of the ellipsoid at H_1 (see Figure 1) and P_2. This plane will intersect the local geodetic horizon plane in a line having an angle with the north axis, which is the direction of the meridian at P_1. This angle is the azimuth of the normal section plane $P_1 - P_2$ denoted as A_{12} and will have components ΔE and ΔN in the local geodetic horizon plane. From plane geometry

$$\tan A_{12} = \frac{\Delta E}{\Delta N} \tag{8}$$

By inspection of equations (6) and (7) we may write the equation for normal section azimuth between points P_1 and P_2 as

$$\tan A_{12} = \frac{\Delta E}{\Delta N} = \frac{-\Delta X \sin \lambda_1 + \Delta Y \cos \lambda_1}{-\Delta X \sin \phi_1 \cos \lambda_1 - \Delta Y \sin \phi_1 \sin \lambda_1 + \Delta Z \cos \phi_1} \tag{9}$$

where $\Delta X = X_2 - X_1$, $\Delta Y = Y_2 - Y_1$ and $\Delta Z = Z_2 - Z_1$