J Geod (2010) 84:481–489
DOI 10.1007/s00190-010-0385-3

ORIGINAL ARTICLE

Global optimization of the Gauss conformal mappings of an ellipsoid to a sphere

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Received: 21 October 2009 / Accepted: 26 April 2010 / Published online: 14 May 2010

Abstract The Gauss conformal mappings (GCMs) of an oblate ellipsoid of revolution to a sphere are those that transform the meridians into meridians, and the parallels into parallels of the sphere. The infinitesimal-scale function associated with these mappings depends on the geodetic latitude and contains three parameters, including the radius of the sphere. Gauss derived these constants by imposing local optimum conditions on certain parallel. We deal with the problem of finding the constants to minimize the Chebyshev or maximum norm of the logarithm of the infinitesimal-scale function on a given ellipsoidal segment (the region contained between two parallels). We show how to solve this minimax problem using the intrinsic function *fminsearch* of Matlab. For a particular ellipsoidal segment, we get the solution and show the alternation property characteristic of best Chebyshev approximations. For a pair of points relatively close in the ellipsoid at different latitudes, the best minimax GCM on the segment defined by these points is used to approximate the geodesic distance between them by the spherical distance between their projections on the corresponding sphere. This approach, combined with the best locally GCM if the points are on the same parallel, is illustrated by applying it to some case studies but specially to a $10° \times 10°$ region contained between portions of two parallels and two meridians. In this case, the maximum absolute error of this spherical approximation is equal to 2.9 mm occurring at a distance about 1,360 km. This error decreases up to 0.94 mm on an $8° \times 8°$ region of this type. So, the spherical approximation to the solution of the inverse geodesic problem by best GCM can be acceptable in many practical geodetic activities.

Keywords Geometrical geodesy · Conformal mapping · Distortion analysis · Inverse geodesic problem

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1 Introduction

In 1843 and 1846, Gauss published two memoirs on topics in higher geodesy under the common title *Untersuchungen über Gegenstände der höhern Geodaesie* (Bühler 1981; Scholz 1992). In the first memoir (Gauss 1843), with the intention of using spherical trigonometry for the purposes of geometrical geodesy, Gauss maps conformally the Earth ellipsoid upon a sphere, whose radius R must be suitably selected. He specifically considers the set of the conformal mappings of an oblate ellipsoid of revolution to a sphere with the property that meridians and parallels of the ellipsoid correspond, respectively, to meridians and parallels of the sphere.

Let σ be the infinitesimal-scale function associated with these conformal mappings, defined as the ratio $\sigma = \mathrm{d}s_\mathrm{s}/\mathrm{d}s$ of the linear element of the sphere $\mathrm{d}s_\mathrm{s}$ to the linear element of the ellipsoid $\mathrm{d}s$ at corresponding points (here and subsequently, we use the subscript 's' to refer to elements of the sphere). This is a function of the geodetic latitude φ and depends on three constants: $\mathbf{c} = (c_1, c_2)$ and $k = \log(R/a)$, where a is the semimajor axis of the given ellipsoid. If φ_0 is the latitude of a parallel passing, for example, centrally through the area to be mapped, Gauss derives the value of the constants $\mathbf{c}$ and k from the local optimal conditions at

$$\sigma = 1, \quad \frac{\mathrm{d}\sigma}{\mathrm{d}\varphi} = 0, \quad \frac{\mathrm{d}^2\sigma}{\mathrm{d}\varphi^2} = 0, \tag{1}$$

at $\varphi = \varphi_0$. For further reading, we refer to Darboux (1972) or Thompson (1975). For abbreviation, we call $(\mathbf{c}, k)$ the *Gauss constants with standard parallel* $\varphi = \varphi_0$.

The purpose of this paper is to get the three constants of the GCMs by global optimization. More exactly, for a given interval

$$\Omega = [\varphi_s, \varphi_n] \subset (-\pi/2, \pi/2), \tag{2}$$

we wish to find $\mathbf{c}$ and k to minimize

$$\max_{\varphi\in\Omega} |\log\sigma(\mathbf{c}, k, \varphi)|. \tag{3}$$

This is a minimax optimization problem that we solve numerically using the Matlab intrinsic function **fminsearch** (The Mathworks Inc. 2009a). We call the solution of this problem *optimal Gauss constants associated with* $[\varphi_s, \varphi_n]$.

In Sect. 2, we define the GCMs. Since we make use of the Gauss constants with standard parallel $\varphi_0 = (\varphi_s + \varphi_n)/2$ to solve the minimax problem (3), Sect. 2 also provides a detailed exposition of the Gauss approach to get the constants $(\mathbf{c}, k)$ from Eq. (1), thus making the paper self-contained. To simplify their derivation, we use the isometric latitude and the Gaussian curvature equation (Chang Sun-Yung 2004). Section 3 is devoted to the study of the minimax problem (3). If h denotes the logarithm of the radius of the parallels in the ellipsoid, we show that this problem is basically reduced to find the solution f of a second-order nonlinear ordinary differential equation to minimize the oscillation of $f - h$. As a particular example, we solve the minimax problem (3) for the ellipsoidal segment $[40°, 50°]$ and show the alternation property characteristic of best Chebyshev approximations.

The rationale for the present study is our interest in obtaining, with a reasonable degree of accuracy, simple approximate formulas for the computation of the geodesic distance between two points in the ellipsoid. In respect of this question, in Sect. 4 we give examples in which we compare the geodesic distance between points in the ellipsoid using Vincenty's (1975) formulas with the geodesic distance between the projected points in the sphere according to the best minimax GCM on the ellipsoidal segment that the points define if their latitudes are different. If the latitudes are equal, then we use the GCM with standard parallel the common one. For a validation of Vincenty's formulas see Thomas and Featherstone (2005). In particular, we show that for the set D of all points (λ, φ), λ being the longitude, where $\lambda \in [0°, 10°]$ and $\varphi \in [40°, 50°]$ (briefly $D = [0°, 10°] \times [40°, 50°]$), the maximum absolute error is 2.9 mm, whereas in $D = [0°, 8°] \times [40°, 48°]$ the maximum absolute deviation is less than 1 mm. All the numerical computations are based on the GRS80 ellipsoid (Moritz 1980).

2 Gauss conformal mappings of an oblate ellipsoid of revolution on a sphere

Let (λ, q) denote the longitude and the isometric latitude of a point in the ellipsoid. The isometric latitude q is related with the geodetic latitude φ by

$$q(\varphi) = \log\left[\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\left(\frac{1 - e\sin\varphi}{1 + e\sin\varphi}\right)^{e/2}\right], \tag{4}$$

where e is the first eccentricity of the ellipsoid. Using these coordinates the linear element of the ellipsoid takes the form

$$\mathrm{d}s^2 = a^2 r^2 (\mathrm{d}\lambda^2 + \mathrm{d}q^2) \tag{5}$$

where $r = r(q)$ is the radius of the parallel of constant isometric latitude q on the similar ellipsoid with $a = 1$. The function $r(q)$ is not exactly known except for the unit sphere, where $r_s = \operatorname{sech} q$ and q is the isometric latitude on the sphere given by Eq. (4) with $e = 0$. However, we have

$$r(q(\varphi)) = \frac{\cos\varphi}{(1 - e^2\sin^2\varphi)^{1/2}}. \tag{6}$$

The GCMs of the ellipsoid to a sphere of radius R are defined by

$$\lambda' = c_1\lambda, \quad q' = c_1 q + c_2 \tag{7}$$

where (λ', q') are the longitude and the isometric latitude of the projected point in the sphere; and where c_1 and c_2 are real constants. These mappings transform the meridians into meridians, and the parallels into parallels of the sphere. Conversely, any (orientation preserving) conformal mapping with this property is of the form (7) if, in addition, the meridian $\lambda = 0$ is transformed into the meridian $\lambda' = 0$. Without loss of generality we can assume $c_1 > 0$ so the image of the North pole ($q = +\infty$) is the North pole of the sphere ($q' = +\infty$).

To get the infinitesimal-scale function σ associated with any of these conformal mappings, we note that

$$\mathrm{d}s_s^2 = R^2 r_s^2(\mathrm{d}\lambda'^2 + \mathrm{d}q'^2) = R^2 r_s^2 c_1^2(\mathrm{d}\lambda^2 + \mathrm{d}q^2) \tag{8}$$

$$= (R/a)^2(c_1 r_s/r)^2 \mathrm{d}s^2. \tag{9}$$

where $\mathrm{d}s_s$ is the linear element of the sphere and $r_s = \operatorname{sech} q'$. Therefore,

$$\sigma = \frac{\mathrm{d}s_s}{\mathrm{d}s} = \exp(k)\frac{c_1\operatorname{sech}(c_1 q + c_2)}{r}, \tag{10}$$

where $k = \log(R/a)$ and exp stands for the exponential function. Writing $g = \log\sigma$, we have

$$g = k + \log[c_1\operatorname{sech}(c_1 q + c_2)] - \log r. \tag{11}$$

Since $\mathrm{d}r/\mathrm{d}q = -r \sin\varphi$ the first derivative of g with respect to q is given by

$$\frac{\mathrm{d}g}{\mathrm{d}q} = -c_1 \tanh(c_1 q + c_2) + \sin\varphi. \tag{12}$$

Remark 2.1 The function g in Eq. (11) satisfies the following second-order ordinary differential equation

$$\frac{\mathrm{d}^2 g}{\mathrm{d}q^2} + r^2 \exp(2g - 2k) = r^2 K, \tag{13}$$

where K is the Gaussian curvature of the ellipsoid (with $a = 1$). Equation (13) is a particular case of the so-called *Gaussian curvature equation* (see Eq. (15) below). In fact, let $p : S \to S'$ be a conformal mapping between two regular surfaces S and S'. Using isometric coordinates (u, v), the linear element of S takes the form

$$\mathrm{d}s^2 = m^2(\mathrm{d}u^2 + \mathrm{d}v^2), \tag{14}$$

where $m = m(u, v)$, and the infinitesimal-scale function associated with p satisfies the differential equation (Chang Sun-Yung 2004)

$$\Delta \log\sigma = m^2(K - \sigma^2 K'), \tag{15}$$

where K and K' are the Gaussian curvatures of S and S', respectively, and $\Delta = \partial^2/\partial u^2 + \partial^2/\partial v^2$ is the 2D Laplace operator. In our case, $m = ar$ by Eq. (5) and $K' = R^{-2}$ is the Gaussian curvature of the sphere. Since g does not depend on the longitude, we conclude from Eq. (15) that g satisfies Eq. (13).

Remark 2.2 A proof of the Gaussian curvature equation (15) is based on the following Gauss' formula for the Gaussian curvature of a surface referred to an isometric system of coordinates (u, v) (Dombrowski 1979)

$$K = -\frac{1}{m^2}\,\Delta \log m. \tag{16}$$

Indeed, if $p : S \to S'$ is a conformal mapping between S and S' and σ is the infinitesimal-scale function associated with p, then the linear element of S' can be written

$$\mathrm{d}s'^2 = \sigma^2 \mathrm{d}s^2 = \sigma^2 m^2(\mathrm{d}u^2 + \mathrm{d}v^2). \tag{17}$$

From this and Eq. (16), the Gaussian curvature K' of S' is given by

$$K' = -\frac{1}{\sigma^2 m^2}\,\Delta \log(\sigma m). \tag{18}$$

Hence

$$\Delta \log\sigma = -m^2\sigma^2 K' - \Delta \log m = m^2(K - \sigma^2 K'), \tag{19}$$

by Eq. (16).

To fix the constants $\mathbf{c}$ and k, Gauss imposes the conditions

$$\sigma = 1, \quad \frac{\mathrm{d}\sigma}{\mathrm{d}\varphi} = 0, \quad \frac{\mathrm{d}^2\sigma}{\mathrm{d}\varphi^2} = 0 \tag{20}$$

on a selected parallel $\varphi = \varphi_0$. It is easy to check that these conditions are equivalent to

$$g = 0, \quad \frac{\mathrm{d}g}{\mathrm{d}q} = 0, \quad \frac{\mathrm{d}^2 g}{\mathrm{d}q^2} = 0 \tag{21}$$

at $q = q_0$, where $q_0 = q(\varphi_0)$ is the isometric latitude corresponding to φ_0. Hence, by Eqs. (11)–(13), we have

$$c_1 \operatorname{sech}(c_1 q_0 + c_2) = \exp(-k) r_0 \tag{22}$$
$$c_1 \tanh(c_1 q_0 + c_2) = \sin\varphi_0 \tag{23}$$
$$\exp(-2k) = K_0, \tag{24}$$

Here r_0 is the radius of the parallel φ_0 and K_0 is the Gaussian curvature of the ellipsoid (with $a = 1$) at $\varphi = \varphi_0$, i.e.,

$$K_0 = \frac{(1 - e^2 \sin^2\varphi_0)^2}{1 - e^2}. \tag{25}$$

Recalling that $k = \log(R/a)$, Eq. (24) gives the radius of the sphere:

$$R = a K_0^{-1/2}. \tag{26}$$

In Mathematical Geodesy, this radius is known to be the Gaussian mean radius at φ_0: $R = \sqrt{M_0 N_0}$, where M_0 is the radius of curvature in the meridian and N_0 is the radius of curvature in the prime vertical.

Since $\tanh^2 x + \operatorname{sech}^2 x = 1$, from Eqs. (22)–(24) we have

$$c_1^2 = \exp(-2k) r_0^2 + \sin^2\varphi_0 = K_0 r_0^2 + \sin^2\varphi_0. \tag{27}$$

After some algebraic manipulation we obtain

$$c_1 = \left(1 + \varepsilon^2 \cos^4\varphi_0\right)^{1/2}, \tag{28}$$

where $\varepsilon^2 = e^2(1 - e^2)^{-1}$ is the square of the second eccentricity of the ellipsoid. Note that $c_1 > 1$. Therefore, in Eq. (7) we have $|\lambda'| > |\lambda|$ and hence "*there is an overlap for sufficiently large values of λ when the ellipsoid is represented on the sphere*" (Thompson 1975): fix $\lambda_1 \in (\pi/c_1, \pi)$ and let $\lambda_2 = \lambda_1 - 2\pi/c_1$; then the points (λ_1, φ) and (λ_2, φ) have the same image on the sphere. Finally, from Eq. (23) we find

$$c_2 = \operatorname{arctanh}(\sin\varphi_0/c_1) - c_1 q_0. \tag{29}$$

The constant c_2 is well defined because $|\sin\varphi_0/c_1| < 1$.

Example 2.1 For $\varphi_0 = 45°$ we get the values given in Table 1. In addition, Fig. 1 displays the function $|\sigma - 1|$. It is worth noting that the conformal mapping is almost perfect ($\sigma \approx 1$) in the ellipsoidal segment centered at $\varphi_0 = 45°$ and 2° width. This is made clear with the following example. For the pair of points ($\lambda = 0°, \varphi = 44°$), ($\lambda = +10°, \varphi = 46°$), the difference between the geodesic distance and the spherical distance between the projected points is equal to 0.445 mm.

Table 1 Gauss constants with standard parallel $\varphi_0 = 45°$

c_1	c_2 $(\times 10^{-3})$	k $(\times 10^{-6})$
1.0008420825454	2.81185202532097	−5.63956226753395

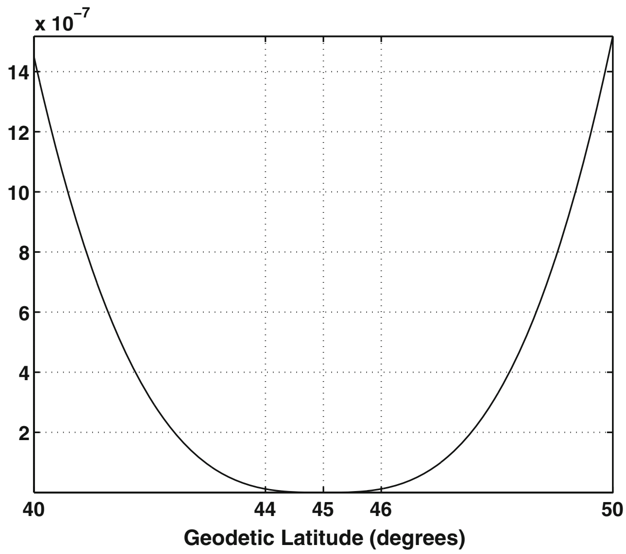

Fig. 1 Absolute deviation from unity of the infinitesimal-scale function associated with the GCM with standard parallel $\varphi_0 = 45°$

3 Minimax optimization of the Gauss conformal mappings

With the notation

$$f(\mathbf{c}, q) = \log[c_1 \operatorname{sech}(c_1 q + c_2)] \tag{30}$$

and $h = \log r$, we write the function $g = \log \sigma$ in Eq. (11) in the form

$$g(\mathbf{c}, k, q) = k + [f(\mathbf{c}, q) - h(q)]. \tag{31}$$

For particular $\mathbf{c}$ and k, let

$$\delta_g(\mathbf{c}, k) = \operatorname*{osc}_{\Omega} \; g(\mathbf{c}, k, q) \tag{32}$$

be the oscillation of the function g in the ellipsoidal segment $\Omega = [\varphi_s, \varphi_n]$, where $q = q(\varphi)$ is given by Eq. (4). In other words, $\delta_g(\mathbf{c}, k)$ is the difference between the maximum $M(\mathbf{c}, k)$ and the minimum $m(\mathbf{c}, k)$ values of the function $g(\mathbf{c}, k, q)$ in Ω. Note that $\delta_g(\mathbf{c}, k)$ does not depend on k. Indeed, we have

$$M(\mathbf{c}, k) = k + \max_{\Omega} [f(\mathbf{c}, q) - h(q)] \tag{33}$$

and

$$m(\mathbf{c}, k) = k + \min_{\Omega} [f(\mathbf{c}, q) - h(q)]. \tag{34}$$

Thus,

$$\delta_g(\mathbf{c}, k) = M(\mathbf{c}, k) - m(\mathbf{c}, k) = \operatorname*{osc}_{\Omega} [f(\mathbf{c}, q) - h(q)]. \tag{35}$$

Therefore, we define the function

$$\delta \colon \mathbb{R}^2 \to \mathbb{R}^+, \quad \mathbf{c} \mapsto \operatorname*{osc}_{\Omega} [f(\mathbf{c}, q) - h(q)]. \tag{36}$$

Theorem 3.1 *Let* $\mathbf{c}^*$ *be the solution of the minimum problem*

$$\delta(\mathbf{c}^*) \le \delta(\mathbf{c}) \quad \textit{for all } \mathbf{c}, \tag{37}$$

and let

$$k^* = -\frac{1}{2}\left(\max_{\Omega} [f(\mathbf{c}, q) - h(q)] + \min_{\Omega} [f(\mathbf{c}, q) - h(q)]\right). \tag{38}$$

Then,

$$\max_{\Omega} |g(\mathbf{c}^*, k^*, q)| \le \max_{\Omega} |g(\mathbf{c}, k, q)| \tag{39}$$

for all $\mathbf{c}$ *and* k.

Proof We have

$$M(\mathbf{c}^*, k^*) = k^* + \max_{\Omega} [f(\mathbf{c}^*, q) - h(q)] = \frac{1}{2}\delta(\mathbf{c}^*) \tag{40}$$

and

$$m(\mathbf{c}^*, k^*) = k^* + \min_{\Omega} [f(\mathbf{c}^*, q) - h(q)] = -\frac{1}{2}\delta(\mathbf{c}^*). \tag{41}$$

Consequently,

$$\max_{\Omega} |g(\mathbf{c}^*, k^*, q)| = \frac{1}{2}\delta(\mathbf{c}^*). \tag{42}$$

On the other hand, if there exist constants $\mathbf{c}$ and k such that

$$\max_{\Omega} |g(\mathbf{c}, k, q)| < \frac{1}{2}\delta(\mathbf{c}^*) \tag{43}$$

then we would have

$$\delta_g(\mathbf{c}, k) < \delta(\mathbf{c}^*). \tag{44}$$

Since $\delta_g(\mathbf{c}, k) = \delta(\mathbf{c})$ this is in contradiction with (37).

Remark 3.1 Problem (37) may be seen as an approximation problem. The function to be approximated is the function h, which satisfies the differential equation

$$\frac{\mathrm{d}^2 h}{\mathrm{d}q^2} + K \exp(2h) = 0, \tag{45}$$

where K is the Gaussian curvature of the ellipsoid with $a = 1$. The function K is close to 1: for example, the maximum and minimum values of K are, respectively, 1.0067 and 0.9933. Note that since K is positive, the function h is concave. Equation (45) is a consequence of the Gauss formula (16) with $u = \lambda$, $v = q$ and $m = r(q)$. On the other hand, the approximant functions $f(\mathbf{c}, q)$ are the solutions of the differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}q^2} + \exp(2f) = 0 \tag{46}$$

Let M denote the set of all these functions. Any element of M is also a concave function. Problem (37) can then be reformulated as follows: given a solution h of (45), find f^* belonging to M such that

$$\operatorname*{osc}_{\Omega}(f^* - h) \leq \operatorname*{osc}_{\Omega}(f - h) \tag{47}$$

for all f in M. Equation (46) is a particular case of the Gaussian curvature equation (15): indeed, $\exp(f)$ is the infinitesimal-scale function of the conformal mapping of the (λ, q) plane $(m = 1, K = 0)$ upon the unit sphere $(K' = 1)$ defined by (7): to the point of the plane (λ, q) corresponds the point of the unit sphere whose isometric coordinates (λ', q') are given by $\lambda' = c_1\lambda$ and $q' = c_1 q + c_2$.

According to Theorem 3.1, to solve the minimax problem (3), i.e., to find $(\mathbf{c}, k)$ to minimize

$$\max_{\Omega} |g(\mathbf{c}, k, q)|, \tag{48}$$

the difficulty is in solving the problem (37). We have not an analytical expression for $\delta(\mathbf{c})$. For this reason, we approximate the solution of the problem (37) by the solution of the corresponding discrete problem, i.e., find $\mathbf{c}$ to minimize

$$\max_{j} [f(\mathbf{c}, q_j) - h(q_j)] - \min_{j} [f(\mathbf{c}, q_j) - h(q_j)], \tag{49}$$

where $q_j = q(\varphi_j)$ and φ_j, $j = 1, \ldots, N$, is a set of points in the interval $\Omega = [\varphi_s, \varphi_n]$ (Watson 1970).

We solve the discrete problem (49) with the function **fminsearch** of Matlab (The Mathworks Inc. 2009a). This function uses the Nelder–Mead method (Nelder and Mead 1965) to find the minimum of a problem specified by:

$$\min_{x} f(x), \tag{50}$$

where x is a vector and $f(x)$ is a function that returns a scalar. The basic syntax is `x=fminsearch(fun,x0)`: **fminsearch** starts at an initial estimate `x0` and attempts to find iteratively a local minimizer `x` of the function `fun` (the algorithm is described in detail in "fminsearch Algorithm" on page 4–11 in The Mathworks Inc. 2009b). Given an ellipsoidal segment $\Omega = [\varphi_s, \varphi_n]$, our minimization problem is

$$\min_{\mathbf{c}} \delta_d(\mathbf{c}), \tag{51}$$

where δ_d is the function

$$\delta_d(\mathbf{c}) = \max_{j} [f(\mathbf{c}, q_j) - h(q_j)] - \min_{j} [f(\mathbf{c}, q_j) - h(q_j)]. \tag{52}$$

As initial estimate $\mathbf{c}_0$ we take the Gauss constants with standard parallel $\varphi_0 = (\varphi_s + \varphi_n)/2$. These constants are given by Eqs. (28) and (29). The Nelder–Mead method does not use numerical or analytic gradients. It is then appropriate to solve the problem (51) because the function $\delta_d(\mathbf{c})$ is not continuously differentiable as we see in Fig. 2. In this figure, it is plotted $\delta_d(\mathbf{c})$ with $c_1 = 1$ on the ellipsoidal segment $\Omega = [40°, 50°]$. It is clear that this function is not differentiable at some point near the origin $c_2 = 0$.

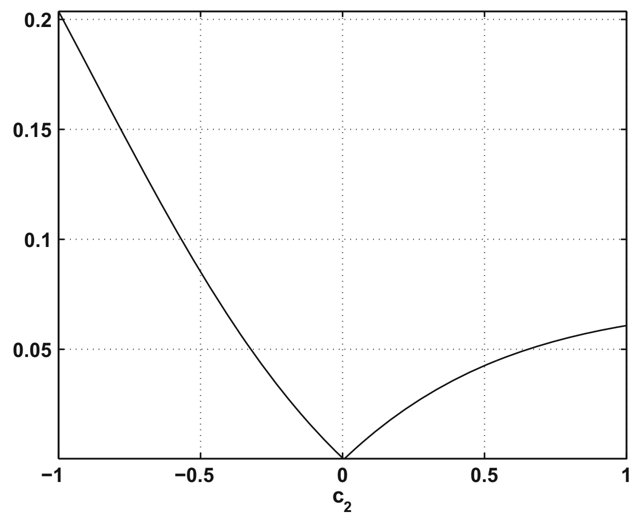

Fig. 2 Plot of the function $\delta_d(1, c_2)$ for $\Omega = [40°, 50°]$

Table 2 Optimal Gauss constants for the ellipsoidal segment [40°, 50°]

c_1^*	c_2^* $(\times 10^{-3})$	k^* $(\times 10^{-6})$
1.00083613843230	2.80741066776071	−6.52822702754130

Example 3.1 In Table 2, we give the components of the vector $\mathbf{c}^*$ and the constant k^*, computed with the Matlab function **fminsearch**, for the ellipsoidal segment [40°, 50°]. The discrete set is $\varphi_{j+1} = \varphi_j + 10^{-2}$, $j = 2, \ldots, N-1$, $\varphi_1 = 40°$ and $\varphi_N = 50°$. We use a termination tolerance on the function value and on $\mathbf{c}$ equal to 10^{-11}: the default values (10^{-4}) are not enough for our purposes because we do not reach the solution.

For this ellipsoidal segment, in Fig. 3 we show the logarithm of the infinitesimal-scale function associated with the best minimax GCM (i.e., the function $g(\mathbf{c}^*, k^*, q)$ where $(\mathbf{c}^*, k^*)$ are given in Table 2). For comparison purpose, we also show $g(\mathbf{c}, k, q)$ where $(\mathbf{c}, k)$ are the Gauss constants with standard parallel $\varphi_0 = 45°$ and whose values are given in Table 1.

We note that the function $g(\mathbf{c}^*, k^*, q)$ alternates three times, i.e., there are four points

$$40° = \varphi_1 < \varphi_2 < \varphi_3 < \varphi_4 = 50° \tag{53}$$

such that

$$g(\mathbf{c}^*, k^*, q_j) = -g(\mathbf{c}^*, k^*, q_{j+1}) = \max_{\Omega} |g(\mathbf{c}^*, k^*, q)|, \tag{54}$$

for all $j = 1, 2, 3$, where $q_j = q(\varphi_j)$. The parallels φ_2 and φ_3 are, respectively, 42.53° and 47.53°. The maximum (or

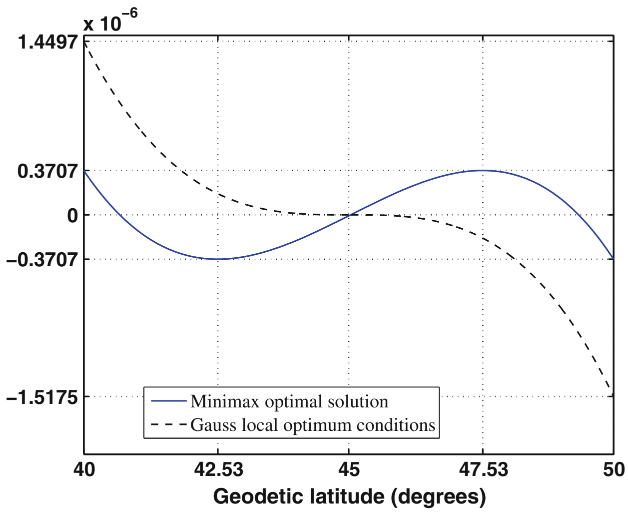

Fig. 3 Logarithm of the infinitesimal-scale function associated with the best minimax GCM on the ellipsoidal segment $[40°, 50°]$

Chebyshev) norm of $g(\mathbf{c}^*, k^*, q)$ in $\Omega = [40°, 50°]$ is equal to 0.3707×10^{-6}:

$$\max_{\Omega} |g(\mathbf{c}^*, k^*, q)| = 0.3707 \times 10^{-6}, \tag{55}$$

whereas

$$\max_{\Omega} |g(c, k, q)| = 1.5175 \times 10^{-6}. \tag{56}$$

The property (54) is characteristic of best Chebyshev approximations (see Rice 1964 for more details).

4 Application to the approximate computation of geodesic distances on the ellipsoid

4.1 A short review of the geodesic inverse problem

Let C be the geodesic line of the ellipsoid passing through the given points $P_1(\lambda_1, \varphi_1)$ and $P_2(\lambda_2, \varphi_2)$. By definition, the distance s_{12} between these points is the arc length of C between them. To appreciate the difficulty of the problem in computing s_{12}, let us briefly describe its fundamentals. For simplicity of exposition we assume that $\varphi_1 < \varphi_2$ and that the angle α (*Azimuth*) that C makes with the meridian at each point between the endpoints inclusive is greater than zero and less than $\pi/2$. Projecting at every point the element of distance ds on the meridian and parallel, we obtain

$$\mathrm{d}s \cos\alpha = r\mathrm{d}q, \tag{57}$$

$$\mathrm{d}s \sin\alpha = r\mathrm{d}\lambda. \tag{58}$$

By the theorem of Clairaut (see for example Struik 1988), along C we have

$$r \sin\alpha = \text{constant}. \tag{59}$$

We denote by c the constant in Eq. (59). From this equation we get

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} = r^{-1}\sqrt{r^2 - c^2}. \tag{60}$$

On substituting (60) into (57) we obtain

$$\mathrm{d}s = r^2(r^2 - c^2)^{-1/2}\,\mathrm{d}q. \tag{61}$$

Hence the distance between the points P_1 and P_2 is given by

$$s_{12} = \int_{q_1}^{q_2} r^2(r^2 - c^2)^{-1/2}\,\mathrm{d}q. \tag{62}$$

In addition, from Eq. (58) we have

$$\mathrm{d}\lambda = c(r^2 - c^2)^{-1/2}\,\mathrm{d}q, \tag{63}$$

by Eqs. (59) and (61). Thus

$$\lambda_2 - \lambda_1 = c\int_{q_1}^{q_2} (r^2 - c^2)^{-1/2}\,\mathrm{d}q. \tag{64}$$

So then the computation of the geodesic distance between points in the ellipsoid requires in principle:

1. To solve in c the nonlinear equation (64) (for example, with the Newton method); and
2. To evaluate numerically the integral in the right-hand side member of Eq. (62).

We wish to approximate s_{12} by the geodesic distance s'_{12} between the projections of P_1 and P_2 on a sphere of radius R following a GCM. If (λ'_i, φ'_i) $(i = 1, 2)$ are the spherical coordinates, longitude and latitude, of the projected points, then s'_{12} is given by the simple formula

$$s'_{12} = R \arccos\left(\sin\varphi'_1 \sin\varphi'_2 + \cos\varphi'_1 \cos\varphi'_2 \cos\Delta\lambda'\right), \tag{65}$$

where $\Delta\lambda' = \lambda'_2 - \lambda'_1$. On the sphere we have $\cos\varphi = \operatorname{sech} q$ and $\sin\varphi = \tanh q$. Therefore, by Eq. (7), we can write

$$s'_{12} = R \arccos[\tanh(c_1 q_1 + c_2)\tanh(c_1 q_2 + c_2) + \operatorname{sech}(c_1 q_1 + c_2)\operatorname{sech}(c_1 q_2 + c_2)\cos(c_1 \Delta\lambda)], \tag{66}$$

where $\Delta\lambda = \lambda_2 - \lambda_1$.

4.2 Numerical examples

We must distinguish two cases: $\varphi_1 = \varphi_2$ and, without loss of generality, $\varphi_1 < \varphi_2$. To evaluate the absolute error we compute the ellipsoidal geodesic distance with the Vincenty formulas (Vincenty 1975): exactly, we use the Matlab program **vdist** written by M. Kleder (http://www.mathworks.com). Hereafter, Δs_{12} denotes the difference $s_{12} - s'_{12}$ for the pair of points $P_1(\lambda_1, \varphi_1)$ and $P_2(\lambda_2, \varphi_2)$.

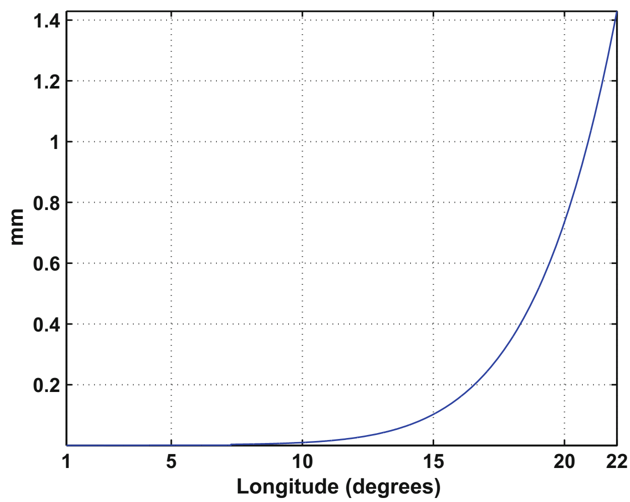

Fig. 4 GCM with standard parallel $\varphi_0 = 45°$: absolute error $|\Delta s_{12}(\lambda)|$ in the spherical approximation of the ellipsoidal geodesic distance between the points $(0°, 45°)$ and $(\lambda, 45°)$

Example 4.1 Let $\varphi_0 := \varphi_1 = \varphi_2$. There is no loss of generality in assuming $\lambda_1 = 0$ and $\lambda_2 \equiv \lambda > 0$. In this situation we approximate s_{12} by s'_{12} where $(\mathbf{c}, R)$ in Eq. (66) are the Gauss constants with standard parallel φ_0. For $\varphi_0 = 45°$ and $\lambda \in [1°, 22°]$, Fig. 4 shows $|\Delta s_{12}(\lambda)|$. We see that the absolute error is less than 0.8 mm if $\lambda \leq 20°$.

If φ_1 and φ_2 are different but close enough each other we can still use the Gauss constants, with standard parallel $\varphi_0 = (\varphi_1 + \varphi_2)/2$, with an admissible accuracy. For example, if $\varphi_1 = 44°$ and $\varphi_2 = 46°$ then $\varphi_0 = 45°$ (see Fig. 1) and the function $|\Delta s_{12}(\lambda)|$ for the pair of points $(0°, 44°) - (\lambda, 46°)$ is less than 0.8 mm if $0° \leq \lambda \leq 12°$.

Example 4.2 In this second example, fix P_1 with geodetic coordinates $\lambda_1 = 0°$ and $\varphi_1 = 40°$. The second point P_2 is on the parallel $\varphi_2 = 50°$ with variable longitude $\lambda_2 \equiv \lambda \in [0°, 10°]$ (see Fig. 5). In this case, the constants $(\mathbf{c}, R)$ appearing in Eq. (66) are the optimal Gauss constants associated with $[40°, 50°]$ (see Table 2). Figure 6 displays the absolute error $|\Delta s_{12}(\lambda)|$. The function $\Delta s_{12}(\lambda)$ is strictly decreasing taking positive values on the interval $[0, 5.75°]$, and hence $|\Delta s_{12}(\lambda)|$ vanishes at $\lambda \approxeq 5.75°$. The global maximum of $|\Delta s_{12}(\lambda)|$ is assumed at $\lambda = 10°$ and is equal to 2.9 mm, approximately.

In Table 3, we summarize the features of $|\Delta s_{12}(\lambda)|$ in $[0°, 10°]$. We note in particular that $|\Delta s_{12}(\lambda)| \leq 1.25$ mm if $\lambda \in [0°, 8°]$.

Using the Gauss constants with standard parallel $\varphi_0 = 45°$, in Fig. 7 it is plotted the absolute error $|\Delta s_{12}(\lambda)|$ for the same pairs of points as above. The function $\Delta s_{12}(\lambda)$ is positive and strictly increasing, and takes its minimum and maximum values at the endpoints of the interval. The minimum value is equal to 7.8 mm, and the maximum one equals

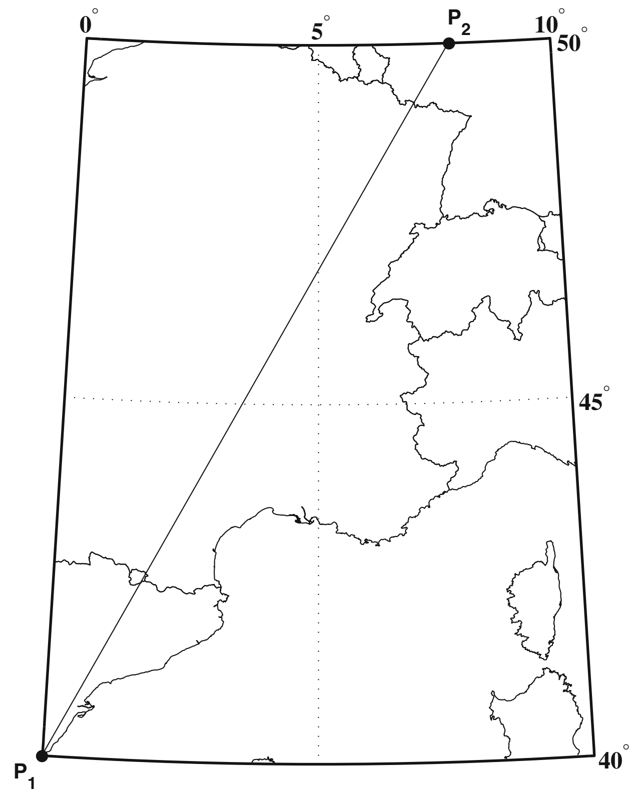

Fig. 5 A map of the rectangular region $D = [0°, 10°] \times [40°, 50°]$ using the Azimuthal Equidistant Projection

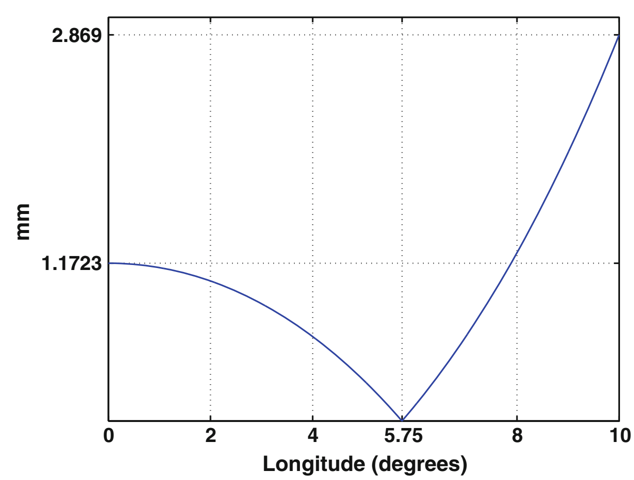

Fig. 6 Best minimax GCM on the ellipsoidal segment $[40°, 50°]$: absolute error $|\Delta s_{12}(\lambda)|$ in the spherical approximation of the ellipsoidal geodesic distance between the points $(0°, 40°)$ and $(\lambda, 50°)$

27.3 mm which is substantially greater than the maximum value of $|\Delta s_{12}(\lambda)|$ corresponding to the best minimax GCM.

Example 4.3 For the whole region, $D = [0°, 10°] \times [40°, 50°]$ in our case, the use of the same set of constants (the Gauss constants with standard parallel $\varphi_0 = 45°$ or the optimal Gauss constants associated with $[40°, 50°]$) for every

Table 3 Relevant values of the function $|\Delta s_{12}(\lambda)|$ plotted in Fig. 6

λ (°)	s_{12} (m)	$\|\Delta s_{12}\|$ (mm)
0	1,111,318.011	1.17
5.75	1,199,329.220	0.00
8	1,276,137.450	1.25
10	1,359,994.883	2.87

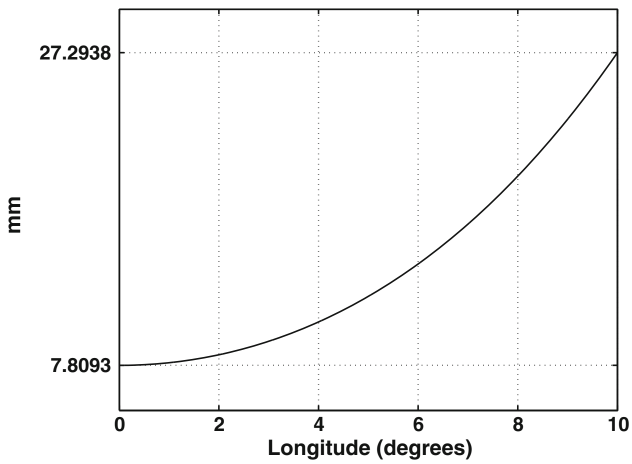

Fig. 7 GCM with standard parallel $\varphi_0 = 45°$: absolute error $|\Delta s_{12}(\lambda)|$ in the spherical approximation of the ellipsoidal geodesic distance between the points $(0°, 40°)$ and $(\lambda, 50°)$

pair of points in D does not give good results. It is better to adapt the GCM for a each particular pair of points. This is clearly seen in Table 4 where we give the maximum absolute error in the spherical approximation of ellipsoidal distances in D, and the pairs of points where these maxima occur, in the following cases:

I Fixed Gauss constants with standard parallel $\varphi_0 = 45°$.
II Fixed optimal Gauss constants associated with [40°, 50°].
III Variable Gauss constants: for the pair of points (λ_i, φ_i), (λ_j, φ_j), with $\varphi_i \leq \varphi_j$, we employ the Gauss constants with standard parallel $\varphi_0 = (\varphi_i + \varphi_j)/2$.
IV Variable optimal Gauss constants: for the pair of points (λ_i, φ_i), (λ_j, φ_j), with $\varphi_i \leq \varphi_j$, we use the optimal Gauss constants associated with $[\varphi_i, \varphi_j]$ if $\varphi_i < \varphi_j$. If the points are on the same parallel then we employ the Gauss constants with standard parallel $\varphi_i = \varphi_j$. If the latitudes are close enough, for example $|\varphi_i - \varphi_j| \leq 2°$ (see Fig. 4), we may use the Gauss constants with standard parallel $\varphi_0 = (\varphi_i + \varphi_j)/2$ to avoid unnecessary optimization.

Errors are computed on an evaluation grid of 21×21 equally spaced points in $D = [0°, 10°] \times [40°, 50°]$.

The results in Table 4 are sorted from the worst to the best case. It is worth noting that in the case II (fixed optimal

Table 4 Maximum absolute errors of the spherical approximation of ellipsoidal geodesic distances between points in the region $[0°, 10°] \times [40°, 50°]$ under different ways of applying the GCMs

Case	Maximum absolute error (m)	Pair of points
I	1.1851	$(0°, 40°) - (10°, 40°)$
II	0.3043	$(0°, 42.5°) - (10°, 42.5°)$
III	0.0273	$(0°, 40°) - (10°, 50°)$
IV	0.0029	$(0°, 40°) - (10°, 50°)$

Gauss constants) the latitude of the pair of points where the maximum absolute error is attained (42.5°) coincides with the latitude 42.53° where $|\log \sigma| \approx |\sigma - 1|$ reaches its first interior maximum value (see Fig. 3). In the cases III and IV, the maximum absolute error is also achieved at the pair of points $(10°, 40°) - (0°, 50°)$.

We expect better accuracy if the region is smaller. For example, if $D = [0°, 8°] \times [40°, 48°]$ then the maximum absolute error in the case IV is equal to 0.94 mm.

5 Conclusions

This paper attempts to be a contribution to the problem, initiated by Gauss, of representing optimally the Earth ellipsoid on a sphere (i.e., as close as possible to an isometry). In its full generality this problem is unsolved. Restricting to conformal mappings, and using the maximum norm of the logarithm of the infinitesimal-scale function as a measure of distortion, we must find a solution of the partial differential equation

$$\Delta f + \exp(2f) = 0 \tag{67}$$

to minimize the oscillation of $f - h$ where $h = \log(r/a)$. Indeed, if g is the logarithm of the infinitesimal-scale function associated with a conformal mapping of the ellipsoid to a sphere, then by the Gaussian curvature equations (15) and (16) (with $m = r$) it is easy to check that $g = f - h + k$ where f is a solution of Eq. (67). As far as we know this approximation problem is unsolved as well.

In the class of the Gauss conformal mappings (GCMs), Δf is simply the second order q-derivative of f and we have showed how to find the best approximation to h using the function **fminsearch** of Matlab. It rests to try to obtain formulas for the optimal constants $(\mathbf{c}^*, k^*)$ (or to write a more specific algorithm to this problem), possibly from the alternation property that the logarithm of the infinitesimal-scale function associated with the best minimax GCM exhibits (see Fig. 3).

If we want to apply the best GCM for computing approximately geodesic distances between two relatively close points in the ellipsoid, we have come to the conclusion that the most suitable is to use the optimal Gauss constants associated with the ellipsoidal segment defined by the two points when their latitudes are different in combination with the Gauss constants if the points are on the same parallel. By doing so we get an acceptable accuracy in cases of practical importance: for example, on the region $[5.8°, 15.1°] \times [47.2°, 55.2°]$, that includes Germany, the maximum absolute error is about 1.47 mm corresponding to the largest distance $1{,}100{,}097.822$ m.

Acknowledgments This research was partially supported by the project MTM 2008-06208 of the Ministerio de Ciencia e Innovación (Spain), and by the BSCH-UCM (Banco Santander Central Hispano and Universidad Complutense de Madrid) Research Groups 910505 and 910480. We are very grateful to Professor Juan Angel García-Martínez (University of Extremadura) for helping us with Matlab programming.

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